

The classification problem for von Neumann factors

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Abstract

We prove that it is not possible to classify separable von Neumann factors of types II_1 , II_∞ or III_λ , $0 \leq \lambda \leq 1$, up to isomorphism by a Borel measurable assignment of “countable structures” as invariants. In particular the isomorphism relation of type II_1 factors is not smooth. We also prove that the isomorphism relation for von Neumann II_1 factors is analytic, but is not Borel.

Key words: Von Neumann algebras; Classification; Borel reducibility; Turbulence.

1 Introduction

The purpose of this paper is to apply the notion of Borel reducibility of equivalence relations, developed extensively in descriptive set theory in recent years, and the deformation-rigidity techniques of Sorin Popa, to study the global structure of the set of factors on a separable Hilbert space.

Recall that if E, F are equivalence relations on standard Borel spaces X and Y , respectively, we say that E is *Borel reducible* to F , written $E \leq_B F$, if there is a Borel $f : X \rightarrow Y$ such that

$$xEy \iff f(x)Ff(y),$$

in other words, if it is possible to classify the points of X up to E equivalence by a Borel assignment of invariants that are F -equivalence classes. We write $E <_B F$ if $E \leq_B F$ but not $F \leq_B E$.

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In this paper, the space of study will be the space $\text{vN}(\mathcal{H})$ of von Neumann algebras on a separable Hilbert space \mathcal{H} . Effros introduced in [2],[3], a standard Borel structure on this space and showed that the set of factors $\mathcal{F}(\mathcal{H})$ is a Borel subset of $\text{vN}(\mathcal{H})$, in particular is a standard Borel space. Let $\simeq^{\text{vN}(\mathcal{H})}$ denote the isomorphism relation in $\text{vN}(\mathcal{H})$ and $\simeq^{\mathcal{F}(\mathcal{H})}$ the isomorphism relation in $\mathcal{F}(\mathcal{H})$. It is natural to ask if it is possible to classify isomorphism classes of factors by an assignment of invariants which are countable groups, countable graphs, countable linear orders, countable fields, or other kinds of “countable structures” type invariants. Using the Axiom of Choice the answer is in principle ‘yes’, given that there are at most continuum many isomorphism classes in \mathcal{F} and continuum many non-isomorphic countable groups. However, this kind of “classification” is of no interest since the Axiom of Choice provides us with no concrete way of computing the invariants. Therefore, the correct question to ask is if $\simeq^{\mathcal{F}}$ is Borel reducible to the isomorphism relation of countable graphs, groups, fields, etc.

To be more specific, one can naturally regard the Polish space

$$\mathbf{GP} = \{(f, e) \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N} : (\forall i, j, k \in \mathbb{N})(f(f(i, j), k) = f(i, f(j, k))) \\ \wedge (\forall i \in \mathbb{N})f(i, e) = f(e, i) = i \wedge (\forall i)(\exists l)f(i, l) = e\}$$

as the set of all countable groups up to isomorphism. The set of countable graphs may naturally be identified with the Polish space

$$\mathbf{GF} = \{g \in \{0, 1\}^{\mathbb{N} \times \mathbb{N}} : (\forall i, j)g(i, j) = g(j, i) \wedge (\forall i)g(i, i) = 0\}.$$

Denote by $\simeq^{\mathbf{GP}}$ the isomorphism relation in \mathbf{GP} , and $\simeq^{\mathbf{GF}}$ the isomorphism relation in \mathbf{GF} . Then we may ask if $\simeq^{\mathcal{F}(\mathcal{H})}$ is Borel reducible to $\simeq^{\mathbf{GP}}$ or $\simeq^{\mathbf{GF}}$.

The appropriate general context to phrase the question is to ask if there is a countable language \mathcal{L} , in the sense of model theory, such that $\simeq^{\mathcal{F}(\mathcal{H})}$ is Borel reducible to the isomorphism relation $\simeq^{\text{Mod}(\mathcal{L})}$ in the space $\text{Mod}(\mathcal{L})$ of countable models of \mathcal{L} (see [9]). If this were the case then we would say that $\simeq^{\mathcal{F}(\mathcal{H})}$ is *classifiable by countable structures*. This generalization encompasses all natural countable structures type invariants, including the two classes \mathbf{GP} and \mathbf{GF} above. Our first result is that the answer is no:

Theorem 1 *The isomorphism relation for factors $\simeq^{\mathcal{F}(\mathcal{H})}$ is not classifiable by countable structures. In fact, the restriction of $\simeq^{\mathcal{F}(\mathcal{H})}$ to any of the classes II_1 , II_∞ and III_λ , $0 \leq \lambda \leq 1$, is not classifiable by countable structures. In particular, the isomorphism relation of type II_1 factors is not smooth.*

Along these lines, in §4 we also prove:

Theorem 2 *The isomorphism relation for injective type III_0 factors is not classifiable by countable structures.*

It should be noted that Woods proved in [26] that isomorphism of ITPFI factors is not smooth in the classical sense of Mackey, Effros and Glimm, by showing that E_0 , defined on $\{0, 1\}^{\mathbb{N}}$ by

$$xE_0y \iff (\exists N)(\forall n \geq N)x(n) = y(n),$$

is Borel reducible to \simeq^{ITPFI} . Our conclusion is much stronger than this. Namely, by a classical result of Baer, E_0 is Borel bi-reducible to the isomorphism relation for countable rank 1 torsion free Abelian groups, and so E_0 classes are essentially countable structures type invariants. It is known that while $E_0 \leq_B \simeq^{\mathbf{GP}}$ and $E_0 \leq_B \simeq^{\mathbf{GF}}$, it also holds that $\simeq^{\mathbf{GP}} \not\leq_B E_0$ and $\simeq^{\mathbf{GF}} \not\leq_B E_0$. The reader is referred to [12] for an introduction to the subject of Borel reducibility.

For our next result, denote by \mathbf{wT}_{ICC} the subset of \mathbf{GP} consisting of countably infinite ICC groups with relative property (T) over an infinite normal subgroup. In §5 we show that the isomorphism relation $\simeq^{\mathbf{wT}_{\text{ICC}}}$ is Borel complete for countable structures, that is, for every countable language \mathcal{L} and every invariant Borel class $\mathcal{C} \subseteq \text{Mod}(\mathcal{L})$ we have $\simeq^{\mathcal{C}} \leq_B \simeq^{\mathbf{wT}_{\text{ICC}}}$. Combining this with a deep result of Popa in [19] we obtain:

Theorem 3 *It holds that $\simeq^{\mathbf{wT}_{\text{ICC}}} <_B \simeq^{\mathcal{F}_{\text{II}_1}}$. Thus $\simeq^{\mathcal{F}_{\text{II}_1}}$ is Borel complete for countable structures, but not classifiable by countable structures.*

It follows from Theorem 3 that, in particular, $\simeq^{\mathbf{GF}} <_B \simeq^{\mathcal{F}_{\text{II}_1}}$ and $\simeq^{\mathbf{GP}} <_B \simeq^{\mathcal{F}_{\text{II}_1}}$. Theorem 3 also has the following consequence:

Corollary 4 *The isomorphism relation $\simeq^{\mathcal{F}}$, regarded as a subset of $\mathcal{F} \times \mathcal{F}$, is complete analytic. In particular, it is not a Borel set.*

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2 The finite case

2.0.0.1 The space $\mathcal{A}(G, X, \mu)$. Let (X, μ) be a standard Borel probability space. Recall that $\text{Aut}(X, \mu)$ is the group of measure preserving trans-

formations of X , which is a Polish group when given the topology it inherits when it is naturally identified with a closed subgroup of the unitary group of $L^2(X, \mu)$. If G is a countable group, we let

$$\mathcal{A}(G, X, \mu) = \{\sigma \in \text{Aut}(X, \mu)^G : (\forall g_1, g_2 \in G) \sigma(g_1 g_2) = \sigma(g_1) \sigma(g_2)\}.$$

It is easy to see that this is a closed subset of $\text{Aut}(X, \mu)^G$ when the latter is given the product topology. We identify $\mathcal{A}(G, X, \mu)$ with the space of measure preserving G -actions on X .

2.0.0.2 The Effros Borel space. Let \mathcal{H} be a separable Hilbert space. We let $\text{vN}(\mathcal{H})$ denote the weakly closed, $*$ -closed subalgebras of $\mathcal{B}(\mathcal{H})$, that is, $\text{vN}(\mathcal{H})$ is the space of von Neumann algebras. Effros has shown that this is a standard Borel space if equipped with the Borel structure generated by the sets

$$\{M \in \text{vN}(\mathcal{H}) : M \cap U \neq \emptyset\}$$

where U is a weakly open subset of $\mathcal{B}(\mathcal{H})$, see [2]. There is a natural Polish topology on $\text{vN}(\mathcal{H})$ called the *Effros-Maréchal* topology. The reader is referred to the detailed study by Haagerup and Winsløw in [7], [8].

2.0.0.3 The group-measure space construction. Let G be a discrete group and (X, μ) a standard Borel probability space. Let $\sigma \in \mathcal{A}(G, X, \mu)$ be a m.p. G -action on X . We define on $L^2(G \times X)$ the unitary operators

$$U_g^\sigma(f)(h, x) = f(g^{-1}h, \sigma(g^{-1})(x)).$$

and for each $\varphi \in L^\infty(X)$ the multiplication operators

$$L_\varphi(f)(h, x) = \varphi(x)f(h, x).$$

We denote by $L^\infty(X) \rtimes_\sigma G$ the finite von Neumann algebra in $\mathcal{B}(L^2(G \times X))$ generated by the U_g , $g \in G$, and L_φ , $\varphi \in L^\infty(X)$.

It is well-known that if σ is ergodic then $L^\infty(X) \rtimes_\sigma G$ is a factor, moreover, if the action is free then $L^\infty(X)$ is a *Cartan subalgebra* of $L^\infty(X) \rtimes_\sigma G$. (Recall that A is a Cartan subalgebra of M if A is a MASA and $\mathcal{N}_M(A)'' = \{u \in \mathcal{U}(M) : uAu^* = A\}'' = M$). By results of Feldman and Moore [4], two free ergodic m.p. actions of possible different groups $\sigma \in \mathcal{A}(G, X, \mu)$ and $\theta \in \mathcal{A}(H, Y, \nu)$ are orbit equivalent if and only if their corresponding inclusions of Cartan subalgebras $L^\infty(X) \subset L^\infty(X) \rtimes_\sigma G$, $L^\infty(Y) \subset L^\infty(Y) \rtimes_\theta H$ are isomorphic. Thus the study of orbit equivalence of m.p. group actions can be translated into a problem regarding inclusions of finite von Neumann algebras. It is worth mentioning that isomorphism between group-measure space von

Neumann algebras does not imply isomorphism of the corresponding Cartan subalgebra inclusions (see [1]). However, Popa's deformation-rigidity machinery [17] assures that this is the case for the particular kinds of groups and group actions we study in this paper.

Lemma 5 *The map $\sigma \mapsto L^\infty(X) \rtimes_\sigma G$ is a Borel function from $\mathcal{A}(G, X, \mu)$ to $\text{vN}(L^2(G \times X))$ when the latter is given the Effros Borel structure.*

PROOF. By the Corollary to Theorem 2 in [2], it is enough to find a countable family $f_n : \mathcal{A}(G, X, \mu) \rightarrow \mathcal{B}(L^2(G \times X))$ of Borel functions such that for each $\sigma \in \mathcal{A}(G, X, \mu)$ we have that $(f_n(\sigma))_{n \in \mathbb{N}}$ is dense in $L^\infty(X) \rtimes_\sigma G$.

Let D_n be an enumeration of the dyadic intervals, and let χ_{D_n} be the characteristic functions. $L^\infty(X)$ is separable in the weak topology and is generated by the functions χ_{D_n} . Thus $L^\infty(X) \rtimes_\sigma G$ is generated by $L_{\chi_{D_n}}$ and U_g^σ , $g \in G$. For each $g \in G$ the function

$$\mathcal{A}(G, X, \mu) \rightarrow \mathcal{B}(L^2(G \times X)) : \sigma \mapsto U_g^\sigma$$

is clearly Borel (in fact continuous) when $\mathcal{B}(L^2(G \times X))$ is given the weak topology, and so are the constant functions

$$\mathcal{A}(G, X, \mu) \rightarrow \mathcal{B}(L^2(G \times X)) : \sigma \mapsto L_{\chi_{D_n}}.$$

Suppose now that $f, g : \mathcal{A}(G, X, \mu) \rightarrow \mathcal{B}(L^2(G \times X))$ are Borel functions. Then for all scalars $r_1, r_2 \in \mathbb{C}$ we have that the functions

$$r_1 f + r_2 g, f^*, f \circ g$$

are Borel. Hence the smallest class of functions containing $\sigma \mapsto U_g^\sigma$ and $\sigma \mapsto L_{\chi_{D_n}}$ and closed under taking linear combinations with rational-complex scalars, adjoint and composition is a countable class of Borel functions $f_n : \mathcal{A}(G, X, \mu) \rightarrow \mathcal{B}(L^2(G \times X))$, $n \in \mathbb{N}$, and for each $\sigma \in \mathcal{A}(G, X, \mu)$ clearly $(f_n(\sigma))_{n \in \mathbb{N}}$ is dense in $L^\infty(X) \rtimes_\sigma G$. \square

Let $\sigma : \mathbb{F}_2 \curvearrowright \mathbb{T}^2 = X$ be the usual action on \mathbb{F}_2 on \mathbb{T}^2 , when \mathbb{F}_2 is viewed as a (finite index) subgroup of $\text{SL}(2, \mathbb{Z})$. It was shown in [25, §3] that there is a dense G_δ set $\text{Ext}(\sigma) \subseteq \text{Aut}(X, \mu)$ extending the action σ to an a.e. free \mathbb{F}_3 -action, specifically, if $\mathbb{F}_2 = \langle a, b \rangle$ and $T_a, T_b \in \text{Aut}(X, \mu)$ are the transformation corresponding to the generators, then the set

$$\text{Ext}(\sigma) = \{S \in \text{Aut}(X, \mu) : T_a, T_b \text{ and } S \text{ generate an a.e. free action of } \mathbb{F}_3\}$$

is a dense G_δ set. For each $S \in \text{Ext}(\sigma)$ we denote by σ_S the corresponding \mathbb{F}_3 action. It is easy to verify that $S \mapsto \sigma_S$ is a continuous map $\text{Ext}(\sigma) \rightarrow$

$\mathcal{A}(\mathbb{F}_3, X, \mu)$. It was shown in [25, §5] that the relation

$$S_1 \sim_{oe} S_2 \iff \sigma_{S_1} \text{ is orbit equivalent to } \sigma_{S_2}$$

has meagre classes, that there is a co-meagre set of dense classes, and that $E_0 \leq_B \sim_{oe}$. The following strengthening was also noted by Kechris in [13, Theorem 17.1]:

Lemma 6 *\sim_{oe} is not classifiable by countable structures.*

PROOF. Let $[E_\sigma]$ denote the full group of the \mathbb{F}_2 -action σ . As noted in [25, p. 280], the conjugation action of $[E_\sigma]$ on $\text{Aut}(X, \mu)$ preserves \sim_{oe} . Since σ is ergodic there is an ergodic m.p. \mathbb{Z} -action on X such that $E_\mathbb{Z} \subseteq E_\sigma$. By [5, Theorem 11 and Claim 13], the conjugation action of $[E_\mathbb{Z}]$ on $\text{Aut}(X, \mu)$ is turbulent. Hence \sim_{oe} has meagre classes, has a dense class and contains a turbulent action, so it follows from [9, Theorem 3.18] that it is generically S_∞ -ergodic, in particular, it is not classifiable by countable structures. \square

Theorem 7 *The von Neumann II_1 factors that arise in the group measure construction from an ergodic a.e. free \mathbb{F}_3 action are not classifiable by countable structures.*

PROOF. It suffices to show that for $S_1, S_2 \in \text{Ext}(\sigma)$, $L^\infty(X) \rtimes_{\sigma_{S_1}} \mathbb{F}_3$ is isomorphic to $L^\infty(X) \rtimes_{\sigma_{S_2}} \mathbb{F}_3$ if and only if σ_{S_1} and σ_{S_2} are orbit equivalent. Namely, if we can do this then by the previous Lemmas the map

$$S \mapsto L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$$

is a Borel reduction of \sim_{oe} to isomorphism in $\mathcal{F}_{\text{II}_1}(L^2(\mathbb{F}_3 \times X))$.

Since orbit equivalent actions give rise to isomorphic factors, we only need to verify that the converse holds for the actions of the form σ_S . By Feldman and Moore's Theorem cited above, it is enough to show that an isomorphism of factors of the form $L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$, $S \in \text{Ext}(\sigma)$ gives an isomorphism of the corresponding Cartan inclusions $L^\infty(X) \subset L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$. This follows from the seminal work of Popa [17] on II_1 factors with trivial fundamental group¹. Indeed, since \mathbb{F}_3 has the Haagerup compact approximation property, by proposition 3.1 in [17], we have that for every $S \in \text{Ext}(\sigma)$, $L^\infty(X) \subset L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$ has the relative property H. Since $L^\infty(X) \subset L^\infty(X) \rtimes_{\sigma} \mathbb{F}_2$ is a rigid embedding and $L^\infty(X) \rtimes_{\sigma} \mathbb{F}_2 \subset L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$, then by proposition 4.6 in [17], $L^\infty(X) \subset L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$ is a rigid embedding. Thus $L^\infty(X)$ is an HT_s Cartan subalgebra of $L^\infty(X) \rtimes_{\sigma_S} \mathbb{F}_3$. But then by Corollary 6.5 in [17], any isomorphism between $L^\infty(X) \rtimes_{\sigma_{S_1}} \mathbb{F}_3$ and $L^\infty(X) \rtimes_{\sigma_{S_2}} \mathbb{F}_3$ can be perturbed

¹ We refer to the reader to [17] for the pertinent definitions.

by a unitary to give an isomorphism of the HT Cartan subalgebras inclusions. It follows that σ_{S_1} and σ_{S_2} are orbit equivalent. \square

Corollary 8 *Let \mathcal{L} be a countable language and let $\text{Mod}(\mathcal{L})$ be the Polish space of countable models of \mathcal{L} . Then it is not possible in Zermelo-Fraenkel set theory without the Axiom of Choice to construct a function*

$$f : \mathcal{F}_{\text{II}_1} \rightarrow \text{Mod}(\mathcal{L})$$

such that

$$M_1 \simeq^{\mathcal{F}_{\text{II}_1}} M_2 \iff f(M_1) \simeq^{\text{Mod}(\mathcal{L})} f(M_2).$$

*In particular, it is not possible to construct such function with codomain **GP** or **GF**.*

PROOF. By well-known Theorems of Solovay [22] and Shelah [21], it is consistent with ZF that every function from one Polish space to another is Baire measurable. Hence if a function as above could be constructed in ZF, then by the previous Theorem it would be consistent that a Baire measurable function $\tilde{f} : \text{Aut}(X, \mu) \rightarrow \text{Mod}(\mathcal{L})$ existed such that

$$S_1 \sim_{oe} S_2 \iff \tilde{f}(S_1) \simeq^{\text{Mod}(\mathcal{L})} \tilde{f}(S_2).$$

However, this is not possible since \sim_{oe} is generically S_∞ -ergodic. \square

Remark. Solovay and Shelah's Theorems are true even if we allow the *countable* Axiom of Choice (countable AC) or the principle of Dependent Choices (DC). Thus it holds that no function as in the Corollary can be constructed in ZF+countable AC or ZF+DC.

3 The semifinite and the purely infinite case

In this section we prove the following:

Theorem 9 *The isomorphism relation for factors of types II_∞ and III_λ , $0 \leq \lambda \leq 1$, are not classifiable by countable structures.*

Mimicking the argument given in the type II_1 case, we exhibit concrete examples of families of factors of types II_∞ and III_λ , $0 \leq \lambda \leq 1$, for which \sim_{oe} is Borel reducible to the isomorphism relation in each of the families. For the proof we rely on the main result of the previous section, on a recent theorem of Popa concerning unique tensor product decomposition of McDuff II_1 factors and on Connes and Takesaki cross product characterization of type III factors.

Throughout this section, for each $S \in \text{Ext}(\sigma)$ we denote by M_S the type II_1 factor $L^\infty(X, \mu) \rtimes_{\sigma_S} \mathbb{F}_3$ constructed in the previous section, with R the unique injective factor of type II_1 and with $R_{0,1} = R \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ the unique type II_∞ injective factor. We need this simple lemma, a proof of which can be found in [7, Corollary 3.8].

Lemma 10 *For each $N \in \text{vN}(\mathcal{H})$, the map $\otimes_N : \text{vN}(\mathcal{H}) \rightarrow \text{vN}(\mathcal{H} \otimes \mathcal{H})$ $M \rightarrow M \otimes N$ is Borel.*

PROOF. [Proof of Theorem 9:]

The type II_∞ case: For each $S \in \text{Ext}(\sigma)$, $M_S \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ is a factor of type II_∞ . Assume that Θ is an isomorphism from $M_{S_1} \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ to $M_{S_2} \otimes \mathcal{B}(\ell^2(\mathbb{N}))$. Then M_{S_1} is isomorphic to an amplification of M_{S_2} . Indeed, if $p = \text{Id} \otimes e_{11}$ then $p M_{S_1} \otimes \mathcal{B}(\ell^2(\mathbb{N})) p \cong M_{S_1}$. Since $\Theta(p)$ is a finite projection on the factor $M_{S_2} \otimes \mathcal{B}(\ell^2(\mathbb{N}))$, there exist two finite projections c_1 and d_1 in M_{S_2} and $\mathcal{B}(\ell^2(\mathbb{N}))$ such that $\Theta(p) \sim c_1 \otimes d_1$. Since $M_{S_2} \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ is a properly infinite factor, there exist two families of mutually equivalent and pairwise orthogonal projections (p_n) and (q_n) such that $\sum p_n = \sum q_n = 1$, $p_1 = \Theta(p)$, $q_1 = c_1 \otimes d_1$ and $p_n \sim q_n$. Denote by $u_n \in M_{S_1} \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ the partial isometry such that $p_n = u_n^* u_n$ and $q_n = u_n u_n^*$. Then $u = \sum u_n$ is a unitary and $u p_1 u^* = q_1$. Thus $\text{Ad}(u) \circ \Theta$ implements an isomorphism between $M_{S_1} \cong p M_{S_1} \otimes \mathcal{B}(\ell^2(\mathbb{N})) p$ and $q_1 M_{S_2} \otimes \mathcal{B}(\ell^2(\mathbb{N})) q_1 = M_{S_2}^t$ where $t = \tau_2 \otimes \text{Tr}(q_1)$. By Corollary 8.2 in [17] the ℓ_{HT}^2 Betti numbers of the factors M_S , $S \in \text{Ext}(\sigma)$, are all equal to the Atiyah ℓ^2 -Betti number of \mathbb{F}_3 , thus $\beta_{\text{HT}}^1(M_S) = \beta_1(\mathbb{F}_3) = 2 \neq 0$. Moreover, $\beta_{\text{HT}}^1(M_{S_2}^t) = \beta_{\text{HT}}^1(M_{S_2})/t = 2/t$. We conclude that $t = 1$. Thus the map $\text{Ext}(\sigma) \ni S \mapsto M_S \otimes \mathcal{B}(\ell^2(\mathbb{N}))$, is a Borel reduction from \sim_{oe} to isomorphism of II_∞ factors.

The type III_λ , $0 < \lambda < 1$, case: Let R_λ , $0 < \lambda < 1$, be the unique injective factor of type III_λ and let $R_\lambda = R_{0,1} \rtimes_\theta \mathbb{Z}$ be its discrete decomposition, where $R_{0,1}$ is the injective II_∞ factor. We consider the family of type III_λ factors $M_{S,\lambda} = M_S \otimes R_\lambda$, $S \in \text{Ext}(\sigma)$. Since $M_{S,\lambda} = M_S \otimes (R_{0,1} \rtimes_\theta \mathbb{Z}) = (M_S \otimes R_{0,1}) \rtimes_{\text{Id} \otimes \theta} \mathbb{Z}$, by the uniqueness of the discrete decomposition [23, Theorem XII.2.1] two factors $M_{S_1,\lambda}$ and $M_{S_2,\lambda}$ are isomorphic if and only if their cores $M_{S_1} \otimes R_{0,1}$ and $M_{S_2} \otimes R_{0,1}$ are isomorphic. The same argument as in the semifinite case shows that this occurs if and only if $M_{S_1} \otimes R$ is isomorphic to an amplification of $M_{S_2} \otimes R$, which in turn is isomorphic to $M_{S_2} \otimes R$, since the fundamental group of R is \mathbb{R}_+^* .

For every $S \in \text{Ext}(\sigma)$, M_S is a non- Γ factor. By Theorem 2 in [18], $M_{S_1} \otimes R$ is isomorphic to $M_{S_2} \otimes R$ if and only if M_{S_1} is isomorphic to an amplification of M_{S_2} . As before, this occurs if and only if M_{S_1} is isomorphic to M_{S_2} . Thus

the maps $\text{Ext}(\sigma) \ni S \mapsto M_{S,\lambda}$, $0 < \lambda < 1$, are Borel reductions from \sim_{oe} to isomorphism of III_λ factors.

The type III_1 case: The same argument used in the type III_λ case, $0 < \lambda < 1$, carries on to the III_1 case once we replace Connes discrete decomposition with Takesaki continuous decomposition. Indeed, let R_∞ be the unique injective factor of type III_1 . If ϕ is a faithful semifinite normal weight on R_∞ , $\{\sigma^\phi\}$ denotes the modular automorphism group associated to the weight ϕ . By Takesaki duality, $R_\infty = R_{0,1} \rtimes_{\sigma^\phi} \mathbb{R}$, where $R_{0,1}$ is the injective II_∞ factor. For each $S \in \text{Ext}(\sigma)$ we consider the type III_1 factor $M_{S,1} = M_S \otimes R_\infty$. Since $M_S \otimes R_\infty = M_S \otimes (R_{0,1} \rtimes_{\sigma^\phi} \mathbb{R}) = (M_S \otimes R_{0,1}) \rtimes_{\text{Id} \otimes \sigma^\phi} \mathbb{R}$, the uniqueness of the continuous decomposition [23, Theorems XII.1.1 & XII.1.7] implies that for $S_1, S_2 \in \text{Ext}(\sigma)$, $M_{S_1,1}$ is isomorphic to $M_{S_2,1}$ if and only if their cores $M_{S_1} \otimes R_{0,1}$ and $M_{S_2} \otimes R_{0,1}$ are isomorphic. We proceed as in the previous case to conclude that \sim_{oe} is Borel reducible to isomorphism of III_1 factors.

The type III_0 case: Once one fixes an injective type III_0 factor, one can either use its discrete or continuous decomposition and repeat the same construction as in the cases $0 < \lambda \leq 1$, to exhibit a family of III_0 factors that are not classifiable by countable structures. Rather than adapting here the previous proofs to take into consideration the fact that the core of a III_0 factor is not a factor, in the next section we show instead that isomorphism relation of injective factors is not classifiable by countable structures. \square

4 The injective type III_0 case

Foreman and Weiss showed in [5] that the isomorphism relation of ergodic measure preserving transformations on a standard probability space (Y, ν) is generically turbulent. The goal here is to combine this result with Kreiger's theorem on the classification of Krieger factors to exhibit a family of injective type III_0 factors that are not classifiable by countable structures. In what follows we assume familiarity with the proof of Krieger's Theorem as presented by Takesaki in [24, XVIII.2] and with the notation used there. Recall that in this context we say that two measure class preserving transformations T_1 and T_2 on a standard measure space are conjugate if there exists a *measure class preserving* automorphism θ such that $\theta T_1 = T_2 \theta$.

If $Q \in \text{Aut}(Y, \nu)$ is ergodic, we construct the ergodic flow F_t^Q under the constant ceiling function $r(y) = 1$. More specifically, for each $t \in \mathbb{R}$, $F_t^Q \in \text{Aut}(Y \times [0, 1], \nu \times m)$ is defined as $F_t^Q(y, s) = (Q^n(y), \alpha)$, where $t + s + \alpha$, $n \in \mathbb{Z}$, $0 \leq \alpha < 1$ (see [23, XII.3]). To be consistent with the notation used by

Takesaki, we denote with (Ω, μ) the standard probability space $(Y \times [0, 1], \nu \times m)$. Since Q is measure preserving, so is the flow F_t^Q . Moreover, it can be shown that two flows F_t^Q and $F_t^{\tilde{Q}}$ constructed in this manner are conjugate if and only if the base transformations Q and \tilde{Q} are conjugate by a *measure class preserving* automorphism, [15, p.31]. Since Q and \tilde{Q} are ergodic and measure preserving, it follows that they are in fact conjugate by a *measure preserving* automorphism.

Since the map $\mathbb{R} \times \text{Aut}(Y, \nu) \rightarrow \text{Aut}(\Omega, \mu) : (t, Q) \mapsto F_t^Q$ is Borel, the map

$$\begin{aligned} \text{Aut}(Y, \nu) &\rightarrow \text{Aut}(X \times \Omega \times \mathbb{R}, P \times \mu \times e^{-s}ds) : \\ Q &\mapsto S^Q(x, \omega, r) = (Tx, F_{a(x)}^Q(\omega), r + b(x)) \end{aligned}$$

of [24, Theorem XVIII.2.5] is Borel. Moreover, S^Q is an ergodic transformation whose associated modular flow is F_t^Q . (Observe that since F_t^Q is m.p., $p(t, \omega)$ defined in [24, p.g. 319 (11)] is equal to 1.)

By Krieger's theorem ([24, Theorem XVIII.2.1]), the type III₀ factors $L^\infty(X \times \Omega \times \mathbb{R}, P \times \mu \times e^{-s}ds) \rtimes_{S^Q} \mathbb{Z}$ and $L^\infty(X \times \Omega \times \mathbb{R}, P \times \mu \times e^{-s}ds) \rtimes_{S^{\tilde{Q}}} \mathbb{Z}$ are isomorphic if and only if the associated modular flows F_t^Q and $F_t^{\tilde{Q}}$ are conjugate. Thus the map $Q \mapsto L^\infty(X \times \Omega \times \mathbb{R}, P \times \mu \times e^{-s}ds) \rtimes_{S^Q} \mathbb{Z}$ is a Borel reduction from isomorphism of ergodic measure preserving transformations to isomorphism of injective III₀ factors. We have shown:

Theorem 11 *The isomorphism relation for injective type III₀ factors is not classifiable by countable structures.*

5 Isomorphism is complete analytic

An equivalence relation E is called *Borel complete* for countable structures if for every language \mathcal{L} and any Borel $\mathcal{C} \subseteq \text{Mod}(\mathcal{L})$ invariant under $\simeq^{\text{Mod}(\mathcal{L})}$ we have $\simeq^{\mathcal{C}} \leq_B E$. In particular, if E is Borel complete for countable structures then we can Borel reduce the isomorphism relation of countable graphs, groups, fields, linear orders, etc., to E . In this section we prove:

Theorem 12 *The isomorphism relation $\simeq^{\mathcal{F}_{\text{II}_1}}$ is Borel complete for countable structures.*

In particular, it follows from this and Theorem 7 that $\simeq^{\mathbf{GF}} <_B \simeq^{\mathcal{F}_{\text{II}_1}}$ and $\simeq^{\mathbf{GP}} <_B \simeq^{\mathcal{F}_{\text{II}_1}}$. Since it is known that $\simeq^{\mathbf{GF}}$ and $\simeq^{\mathbf{GP}}$ are complete analytic as subsets of $\mathbf{GF} \times \mathbf{GF}$ and $\mathbf{GP} \times \mathbf{GP}$, respectively, we obtain from Theorem 12 that $\simeq^{\mathcal{F}_{\text{II}_1}}$ is a complete analytic subset of $\mathcal{F}_{\text{II}_1} \times \mathcal{F}_{\text{II}_1}$, see Corollary 15 below.

Let \mathbf{wT}_{ICC} denote the subset of \mathbf{GP} consisting of ICC groups with the relative property (T) over an infinite normal subgroup. It may be verified that \mathbf{wT}_{ICC} is a Borel subset of \mathbf{GP} . To prove Theorem 12 we will use a deep result of Sorin Popa [19, Theorem 7.1] which implies that for groups $G_1, G_2 \in \mathbf{wT}_{\text{ICC}}$ it holds that if the II_1 factors M_{G_1} and M_{G_2} that arise from Bernoulli shifts of G_1 and G_2 via the group measure space construction are isomorphic then the groups G_1 and G_2 are isomorphic.

Lemma 13 *The equivalence relation $\simeq^{\mathbf{wT}_{\text{ICC}}}$ is Borel complete for countable structures.*

PROOF. By a graph we mean a symmetric, irreflexive relation. Graphs are not assumed to be connected unless otherwise stated. All graphs here are countable.

We will modify Mekler's construction in [16]. Recall that Mekler introduces a notion of “nice” graph [16, Defn. 1.1], and shows in [16, §2] that for each nice graph Γ there is an associated nil-2 exponent p group $G(\Gamma)$, which we will call the Mekler group of Γ . As noted in [6], the association $\Gamma \mapsto G(\Gamma)$ is a Borel reduction of isomorphism of nice graphs to isomorphism of nil-2 exponent p groups. Mekler also shows that the isomorphism relation of *connected* nice graphs is Borel complete for countable structures, [16, §1].

Let Γ be a connected nice graph. Define a graph $\Gamma_{\mathbb{F}_2}$ on $\mathbb{F}_2 \times \Gamma$ by

$$(a_1, v_1)\Gamma_{\mathbb{F}_2}(a_2, v_2) \iff a_1 = a_2 \wedge v_1\Gamma v_2.$$

Then $\Gamma_{\mathbb{F}_2}$ is nice (but not connected). The shift action of $\sigma : \mathbb{F}_2 \curvearrowright \Gamma_{\mathbb{F}_2}$ defined by

$$\sigma(a)(b, v) = (a^{-1}b, v)$$

is clearly an action by graph automorphisms of \mathbb{F}_2 on $\Gamma_{\mathbb{F}_2}$. Recall that for a fixed prime p , the associated Mekler group $G(\Gamma_{\mathbb{F}_2})$ is defined as

$$\left(\mathbf{2}_{(a,v) \in \Gamma_{\mathbb{F}_2}} \mathbb{Z}/p\mathbb{Z} \right) / N$$

where $N = \langle [(a_1, v_1), (a_2, v_2)] : (a_1, n_1)\Gamma_{\mathbb{F}_2}(a_2, n_2) \rangle$, and $\mathbf{2}$ denotes the free product in the category of nil-2 exponent p groups. The action σ corresponds to an action of $\sigma_0 : \mathbb{F}_2 \curvearrowright \mathbf{2}_{(a,v) \in \Gamma_{\mathbb{F}_2}} \mathbb{Z}/p\mathbb{Z}$ by group automorphisms in the obvious way. Since σ acts by graph automorphisms, N is σ_0 -invariant. Hence σ_0 factors to an action $\hat{\sigma}$ of \mathbb{F}_2 on $G(\Gamma_{\mathbb{F}_2})$ by group automorphisms, and we may form the semi-direct product $G(\Gamma_{\mathbb{F}_2}) \rtimes_{\hat{\sigma}} \mathbb{F}_2$.

Claim. $G(\Gamma_{\mathbb{F}_2}) \rtimes_{\hat{\sigma}} \mathbb{F}_2$ is ICC.

Proof. Since \mathbb{F}_2 is ICC, it is enough to consider elements of the form $(g, e) \in G(\Gamma_{\mathbb{F}_2}) \rtimes_{\hat{\sigma}} \mathbb{F}_2$, $g \neq e$. Since $(e, a)(g, e)(e, a^{-1}) = (\hat{\sigma}(a)(g), e)$ it is enough to show that

$$[g]_{\hat{\sigma}} = \{\hat{\sigma}(a)(g) : a \in \mathbb{F}_2\}$$

is infinite. Suppose for a contradiction that this set is finite. Let

$$\mathbf{2}\Gamma_{\mathbb{F}_2} = \bigoplus_{(a,v) \in \Gamma_{\mathbb{F}_2}} \mathbb{Z}/p\mathbb{Z}$$

and identify this group with the free nil-2 exponent p group generated by the vertices of the graph $\Gamma_{\mathbb{F}_2}$. Let $\varphi : \mathbf{2}\Gamma_{\mathbb{F}_2} \rightarrow G(\Gamma_{\mathbb{F}_2})$ be the quotient map with $\ker(\varphi) = N$.

If $[g]_{\hat{\sigma}}$ is finite, then we may find finitely many nodes $(a_1, v_1), \dots, (a_k, v_k) \in \Gamma_{\mathbb{F}_2}$ such that each element of $[g]_{\hat{\sigma}}$ can be written as

$$\varphi((a_{i_1}, v_{i_1}) \cdots (a_{i_l}, v_{i_l}))$$

for an appropriate choice of $1 \leq i_1, \dots, i_l \leq k$. Let $\Delta = \{a_1, \dots, a_k\}$ and let $b \in \mathbb{F}_2$ be such that $b^{-1}\Delta \cap \Delta = \emptyset$. Since $\mathbf{2}\Gamma_{\mathbb{F}_2}$ is freely generated, we have a natural homomorphism

$$p_{\Delta} : \mathbf{2}\Gamma_{\mathbb{F}_2} \rightarrow \mathbf{2}\Delta \times \Gamma,$$

under which

$$p_{\Delta}((a_{i_1}, v_{i_1}) \cdots (a_{i_l}, v_{i_l})) = (a_{i_1}, v_{i_1}) \cdots (a_{i_l}, v_{i_l}).$$

Then since

$$\sigma_0(b)((a_{i_1}, v_{i_1}) \cdots (a_{i_l}, v_{i_l})) = (b^{-1}a_{i_1}, v_{i_1}) \cdots (b^{-1}a_{i_l}, v_{i_l})$$

we have $p_{\Delta}(\sigma_0(b)((a_{i_1}, v_{i_1}) \cdots (a_{i_l}, v_{i_l}))) = e$. It follows that if $\hat{\sigma}(b)(g)$ is represented by a reduced word in $(a_1, v_1), \dots, (a_k, v_k)$ it must be the empty word. Hence $\hat{\sigma}(b)(g) = e$ and so $g = e$, a contradiction. [Claim] \square

Define

$$G_{\Gamma} = \mathrm{SL}(3, \mathbb{Z}) \times G(\Gamma_{\mathbb{F}_2}) \rtimes_{\hat{\sigma}} \mathbb{F}_2.$$

Since $\mathrm{SL}(3, \mathbb{Z})$ has property (T) of Kazhdan and is ICC, then $G_{\Gamma} \in \mathbf{wT}_{\mathrm{ICC}}$. We claim that the map $\Gamma \mapsto G_{\Gamma}$ is a Borel reduction of isomorphism of nice connected graphs to $\simeq^{\mathbf{wT}_{\mathrm{ICC}}}$. It is routine to verify that $\Gamma \mapsto G_{\Gamma}$ is Borel. It is also clear that if $\Gamma \simeq \Gamma'$ then $G_{\Gamma} \simeq G_{\Gamma'}$. So suppose that $G_{\Gamma} \simeq G_{\Gamma'}$. Let

$$H_0 = \langle g \in G_{\Gamma} : (\exists \chi \in \mathrm{Char}(G_{\Gamma})) \chi(g) \neq 1 \rangle.$$

It is clear that $\mathbb{F}_2 \subseteq H_0 \subseteq G(\Gamma_{\mathbb{F}_2}) \rtimes_{\hat{\sigma}} \mathbb{F}_2$, since $\mathrm{SL}(3, \mathbb{Z})$ has no non-trivial characters. Since $G(\Gamma_{\mathbb{F}_2})$ is an exponent p group we have

$$G(\Gamma_{\mathbb{F}_2}) = \{g \in G_{\Gamma} : g = e \vee (g^p = e \wedge (\exists h \in H_0) hgh^{-1} \neq g)\}$$

from which it follows that if $G_\Gamma \simeq G_{\Gamma'}$ then

$$G(\Gamma_{\mathbb{F}_2}) \simeq G(\Gamma'_{\mathbb{F}_2}).$$

By [16, Lemma 2.2] it follows that $\Gamma_{\mathbb{F}_2} \simeq \Gamma'_{\mathbb{F}_2}$. Since the connected components of $\Gamma_{\mathbb{F}_2}$ and $\Gamma'_{\mathbb{F}_2}$ are all isomorphic to Γ and Γ' , respectively, it follows that $\Gamma \simeq \Gamma'$.

As noted above, the isomorphism relation of connected nice graphs is Borel complete for countable structures, and so it follows that $\simeq^{\mathbf{wT}_{\text{ICC}}}$ is also Borel complete for countable structures. \square

Theorem 14 *It holds that $\simeq^{\mathbf{wT}_{\text{ICC}}} <_B \simeq^{\mathcal{T}_{\text{II}_1}}$. Hence $\simeq^{\mathcal{T}_{\text{II}_1}}$ is Borel complete for countable structures, but not classifiable by countable structures.*

It is clear that Theorem 14 implies Theorem 12.

Notation: If $G \in \mathbf{GP}$, we will write \cdot_G for composition in G , $^{-1_G}$ for the inverse operation in G , and e_G for the identity in G .

PROOF. [Proof of Theorem 14] Let $X = [0, 1]^{\mathbb{N}}$, equipped with the product measure μ . For each $G \in \mathbf{GP}$ let $\sigma_G : G \curvearrowright X$ be the Bernoulli shift action,

$$\sigma_G(g)(x)(g_0) = x(g^{-1_G} \cdot_G g_0).$$

For each G , let $M_G = L^\infty(X) \rtimes_{\sigma_G} G$.

Claim. For each $g \in \mathbb{N}$, the map $G \mapsto \sigma_G(g)$ is a continuous map $\mathbf{GP} \rightarrow \text{Aut}(X, \mu)$.

Proof of Claim. Fix intervals $I_1, \dots, I_k \subseteq [0, 1]$ and $n_1, \dots, n_k \in \mathbb{N}$. Consider the cylinder set

$$B = \{x \in X : (\forall i \leq k) x(n_i) \in I_i\},$$

and fix $G_0 \in \mathbf{GP}$. Define

$$N(G_0) = \{G \in \mathbf{GP} : (\forall i \leq k) g \cdot_G n_i = g \cdot_{G_0} n_i\}.$$

Clearly $N(G_0)$ is a clopen subset of $\mathbf{GP} \subseteq \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N}$ in the subspace topology.

If $G \in N(G_0)$ then

$$\begin{aligned}
x \in \sigma_G(g)(B) &\iff \sigma_G(g^{-1G})(x) \in B \\
&\iff (\forall i \leq k) \sigma_G(g^{-1G})(x)(n_i) \in I_i \\
&\iff (\forall i \leq k) x(g \cdot_G n_i) \in I_i \\
&\iff (\forall i \leq k) x(g \cdot_{G_0} n_i) \in I_i \\
&\iff (\forall i \leq k) \sigma_{G_0}(g^{-1G_0})(x)(n_i) \in I \\
&\iff x \in \sigma_{G_0}(g)(B)
\end{aligned}$$

Thus for $G \in N(G_0)$ we have $\sigma_G(g)(B) = \sigma_{G_0}(g)(B)$. Since any measurable $C \subseteq X$ may be approximated by a finite union of cylinder sets, it follows from the above that $G \mapsto \sigma_G(g)$ is continuous. [Claim] \square

Using the claim, it is clear that for each $g \in \mathbb{N}$ the map $G \mapsto U_g^{\sigma G}$, where $U_g^{\sigma G}$ is the unitary operator on $L^2(X \times \mathbb{N})$ defined by

$$U_g^{\sigma G}(f)(x, g_0) = f(\sigma_G(g)(x), g^{-1G} \cdot_G g_0),$$

is continuous. Exactly as in Lemma 5 we then have

$$G \mapsto M_G = L^\infty(X) \rtimes_{\sigma_G} G$$

is Borel, since M_G is generated by $U_g^{\sigma G}$, $g \in G$, and L_ψ , $\psi \in L^\infty(X)$. We claim that the map $\mathbf{wT}_{\text{ICC}} \rightarrow \mathcal{F}_{\text{II}_1} : G \mapsto M_G$ is a Borel reduction of $\simeq^{\mathbf{wT}_{\text{ICC}}}$ to $\simeq^{\mathcal{F}_{\text{II}_1}}$. Clearly, if $G_0, G_1 \in \mathbf{wT}_{\text{ICC}}$ are isomorphic then σ_{G_0} and σ_{G_1} are conjugate, and so M_{G_0} and M_{G_1} are isomorphic. On the other hand, if $M_{G_0} \simeq M_{G_1}$ then we may apply Theorem 7.1 in [19] to conclude that $G_0 \simeq G_1$. This together with Theorem 7 proves that $\simeq^{\mathbf{wT}_{\text{ICC}}} <_B \simeq^{\mathcal{F}_{\text{II}_1}}$. \square

Remark. It should be noted that Corollary 7.2 in [19] subsumes that at least one of the groups in question is ICC. Thus it does *not* suffice for Theorem 14 to look at groups of the form $\text{SL}(3, \mathbb{Z}) \times G(\Gamma)$ for some nice graph Γ .

Corollary 15 *The isomorphism relation $\simeq^{\mathcal{F}_{\text{II}_1}}$ is a complete analytic subset of $\mathcal{F}_{\text{II}_1} \times \mathcal{F}_{\text{II}_1}$. In particular, it is not Borel.*

PROOF. Let $\simeq_u^{\mathcal{F}}$ denote unitary equivalence in \mathcal{F} . By [3, Lemma 2.1] the action of $\mathcal{U}(\mathcal{H})$ on \mathcal{F} is Borel, thus $\simeq_u^{\mathcal{F}}$ is analytic. As noted on p. 436 in [3], the map

$$M \mapsto M \otimes \mathbb{C} \in \mathcal{F}(\mathcal{H} \otimes \mathcal{H}),$$

constitutes a Borel reduction of $\simeq^{\mathcal{F}(\mathcal{H})}$ to $\simeq_u^{\mathcal{F}(\mathcal{H} \otimes \mathcal{H})}$, since any algebraic isomorphism between two separable von Neumann algebras with properly infinite commutants is unitarily implemented. It follows that $\simeq^{\mathcal{F}}$ is analytic. To see

that it is complete analytic, note that by the previous Theorem the isomorphism relation of countable Abelian p -groups is Borel reducible to $\simeq^{\mathcal{F}_{\Pi_1}}$. Since this equivalence relation is complete analytic by [6, Theorem 6], it follows that $\simeq^{\mathcal{F}_{\Pi_1}}$ is complete analytic. \square

Remark. It is possible to derive Corollary 15 as a consequence of [10, Corollary 0.7], by showing that the construction given there of a factor M with $\text{Out}(M) \simeq K$ for a prescribed compact Abelian group K is Borel in the codes. However, it is not known if isomorphism of countable Abelian groups is Borel complete for countable structures, so this approach does not give a different proof of Theorem 14.

5.0.0.4 While the results of this paper indicate that the isomorphism relation for factors is, in general, very complicated, we point out that it is not as complicated (from the point of view of \leq_B) as some other analytic equivalence relations. An equivalence relation E on a standard Borel space is *below a group action* if there is a Borel action of a Polish group G on a Polish space X such that $E \leq_B E_G$, where E_G denotes the orbit equivalence relation in X . Then we have:

Theorem 16 (i) *Isomorphism of separable von Neumann factors is below a group action.*

(ii) *If H is a countable group then orbit equivalence of probability measure preserving a.e. free ergodic actions of H is below a group action.*

PROOF. We have already seen in the proof of Corollary 15 that

$$M \mapsto M \otimes \mathbb{C} \in \mathcal{F}(\mathcal{H} \otimes \mathcal{H}),$$

is a Borel reduction of algebraic isomorphism to unitary equivalence, hence (i) is clear.

For (ii), let (X, μ) be a standard Borel probability space and let $\mathcal{H} = L^2(H \times X)$. Let

$$\mathcal{U}_{L^\infty(X) \otimes \mathbb{C}}(\mathcal{H} \otimes \mathcal{H}) = \{u \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) : uL^\infty(X) \otimes \mathbb{C}u^* = L^\infty(X) \otimes \mathbb{C}\}$$

be the stabilizer of $L^\infty(X) \otimes \mathbb{C}$, which is a closed subgroup of $\mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. Let E denote the orbit equivalence relation induced by the conjugation action of $\mathcal{U}_{L^\infty(X) \otimes \mathbb{C}}(\mathcal{H} \otimes \mathcal{H})$ on $\text{vN}(\mathcal{H} \otimes \mathcal{H})$. We claim that orbit equivalence of a.e. free ergodic actions of H on (X, μ) is Borel reducible to E .

For $\sigma \in \mathcal{A}(H, X, \mu)$ let $M_\sigma = L^\infty(X) \rtimes_\sigma H$. Then the map

$$\mathcal{A}(H, X, \mu) \rightarrow \text{vN}(\mathcal{H} \otimes \mathcal{H}) : \sigma \mapsto M_\sigma \otimes \mathbb{C}$$

is Borel, and if $\sigma_0, \sigma_1 \in \mathcal{A}(H, X, \mu)$ are orbit equivalent a.e. free ergodic actions then the inclusions $L^\infty(X) \subset M_{\sigma_0}$ and $L^\infty(X) \subset M_{\sigma_1}$ are isomorphic. It follows that there is a unitary $u \in \mathcal{U}_{L^\infty(X) \otimes \mathbb{C}}$ such that

$$uM_{\sigma_0} \otimes \mathbb{C}u^* = M_{\sigma_1} \otimes \mathbb{C}.$$

If conversely there is a unitary $u \in \mathcal{U}_{L^\infty(X) \otimes \mathbb{C}}$ such that

$$uM_{\sigma_0} \otimes \mathbb{C}u^* = M_{\sigma_1} \otimes \mathbb{C}$$

then we have the isomorphic inclusions

$$\begin{aligned} L^\infty(X) \subset M_{\sigma_0} &\simeq L^\infty(X) \otimes \mathbb{C} \subset M_{\sigma_0} \otimes \mathbb{C} \\ &\simeq L^\infty(X) \otimes \mathbb{C} \subset M_{\sigma_1} \otimes \mathbb{C} \simeq L^\infty(X) \subset M_{\sigma_1}, \end{aligned}$$

and so by Feldman and Moore's Theorem, σ_0 is orbit equivalent to σ_1 . \square

Remark. By [14, Theorem 4.2], it follows from the previous Theorem that the equivalence relation E_1 on $\mathbb{R}^\mathbb{N}$, defined by

$$xE_1y \iff (\exists N)(\forall n \geq N)x(n) = y(n)$$

is not Borel reducible to $\simeq^\mathcal{F}$. In particular, $\simeq^\mathcal{F}$ is not universal for analytic relations, i.e., not every analytic equivalence relation is Borel reducible to $\simeq^\mathcal{F}$. Part (ii) of the previous Theorem answers a question raised by Hjorth.

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